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MOMENTUM–INDEPENDENT RENORMALIZATION OF THE SCHWINGER–DYSON EQUATION WITH AN INSTANTANEOUS–INTERACTION KERNEL

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We consider the problem of renormalization of the Schwinger–Dyson equation, encountered when the interquark interaction kernel is modeled by an instantaneous potential. More precisely, the Coulomb potential, needed for describing the short–distance part of the gluon exchange, leads to ultraviolet divergences in the Schwinger–Dyson equation. The standard prescription which has been used up till now to subtract these divergences, suffers from a serious conceptual problem: it is not momentum independent. In this work we propose a new and momentum–independent renormalization prescription, and take special care to preserve the correct chiral limit for light pseudoscalar mesons.

1. *Introduction*

The dressed quark propagator could in principle be determined by solving the corresponding Schwinger–Dyson equation (SDE). However, such a SDE for a two–point Green function includes a three–point Green function (vertex function), which should in turn be determined from its own SDE, and the latter again couples to still higher SDE, *etc.* ad infinitum. One is thus faced with an infinite set of coupled SDE's, which is impossible to solve in practice. Therefore, any tractable approach must include the truncation of this infinite power of equations. The ladder approximation, where bare vertices are used instead of dressed ones, is the most standard such truncation, *i.e.*, the interaction kernel is assumed to be of the form

$$K(k) = -\mathrm{i} C_F g^2 \gamma^\mu \otimes \gamma^\nu D_{\mu\nu}(k) \;, \tag{1}$$

where C_F is the second Casimir invariant of the quark representation (4/3 for the case of SU(3) triplet), g is the strong–coupling constant, and D is the gluon propagator. A further standard approximation is the instantaneous approximation to the kernel K leading to the potential model:

$$K(k) \approx \mathrm{i}\gamma^0 \otimes \gamma^0 \widetilde{V}(\vec{k}) - \mathrm{i}\gamma^j \otimes \gamma^l \widetilde{V}_T(\vec{k})\,[\delta^{jl} - \frac{k^j k^l}{|\vec{k}|^2}] \;. \tag{2}$$

A still further approximation consists in neglecting the transverse gluon exchange. Thus, the kernel K becomes

$$K(k) \approx \mathrm{i}\gamma^0 \otimes \gamma^0 \widetilde{V}(\vec{k}) \;. \tag{3}$$

We limit ourselves to the line of research (*e.g.*, Refs. 1-8) where a pairing model is studied in the Coulomb gauge using phenomenological static potentials to bind quarkonia and to cause dynamical chiral symmetry breaking. A popular choice has been the Coulomb–plus–linear ("funnel") potential (*e.g.*, [2-7])

$$V(r) = V_C(r) + V_L(r), \quad V_C(r) = -\frac{4}{3}\,\frac{\alpha_s}{r} \;, \quad V_L(r) = \frac{4}{3}\,\sigma\, r, \tag{4}$$

since it incorporates two main features of interquark forces arising from QCD, namely, one-vector-boson (gluon) exchange at short distances and linear confinement at long distances. (In this paper, instantaneous potentials are denoted by the letter V when they are given in the coordinate space, and by $\widetilde{V}$ when they are Fourier-transformed to the momentum space.)

In the ladder approximation, the "dressed" quark propagator $S(p)$ is given by the SDE

$$S^{-1}(p) = \not{p} - m - \mathrm{i} g^2 C_F \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \gamma^\mu S(k) \gamma^\nu D_{\mu\nu}(p-k) \;, \tag{5}$$

which, however, can be divergent, depending on what is the interaction kernel under the integral over momenta k. For the case given by Eqs. (3) and (4), the UV divergences due to the Coulomb part of the potential require renormalization and introduction of counterterms. This will change Eq. (5) and its rest frame version. Following Ref. 2 and its generalizations to the massive case [4,9], we use the equations for renormalized vector, axial–vector and pseudoscalar vertices, and Ward identities, to obtain the renormalized SDE in the ladder approximation,

$$S^{-1}(p) = Z_2 \not{p} - Z_m m - \mathrm{i} g^2 C_F \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \gamma^\mu S(k) \gamma^\nu D_{\mu\nu}(p-k) \ , \tag{6}$$

where Z_2 and Z_m are the wave–function and mass renormalization constants, respectively, defined by

$$S_0 = Z_2 S \quad \text{ and } \quad m_0 = \frac{Z_m}{Z_2} m \ , \tag{7}$$

where S_0 and m_0 are the bare quark propagator and the bare quark mass, respectively. Neglecting the retardation effects, Eq. (2), and the transverse gluon exchange, Eq. (3), from the renormalized SDE (6) we finally obtain its special case with the instantaneous interaction Eq. (4):

$$S^{-1}(p) = Z_2 \not{p} - Z_m m + \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ \gamma^0 S(k) \gamma^0 [\widetilde{V}_C(\vec{p} - \vec{k}) + \widetilde{V}_L(\vec{p} - \vec{k})] \ . \tag{8}$$

Here $\widetilde{V}_{C,L}$ are the Fourier transforms of $V_{C,L}(r)$, and Z_2, Z_m are the wave–function and mass renormalization constants which must cancel the UV divergences caused by the Coulomb part $\widetilde{V}_C$; m is the current–quark mass.

We have some conceptual objections to the renormalization prescriptions, *i.e.*, choices of Z_2 and Z_m which have been used in this line of research (even in the most recent works such as Ref. 7) to remove the UV divergences in (8) stemming from the Coulomb part $\widetilde{V}_C(\vec{p}-\vec{k})$. In this paper, we propose a new and momentum–independent renormalization prescription.

2. *Multiplicative renormalizability*

Let us demonstrate the multiplicative renormalizability (MR) of Eq. (8) The renormalization of the product $g^2 D$ is

$$g_0^2 D_0 = (\frac{Z_1}{Z_2})^2 g^2 D \ , \tag{9}$$

where the subscripts in g_0 and D_0 refer to the bare quantities and Z_1 is the vertex renormalization constant. The gauge invariance implies $Z_1 = Z_2$ and the renormalization–group (RG) invariance of $g^2 D$. However, the ladder approximation is consistent with $Z_1 = 1$, *i.e.*, with no vertex renormalization (see, *e.g.*, Ref. 10). Thus, the renormalization of $g^2 D$, and hence of $\widetilde{V}$ should be

$$g_0^2 D_0 = (\frac{1}{Z_2})^2 g^2 D \ , \tag{10}$$

$$\widetilde{V}_0 = (\frac{1}{Z_2})^2 \widetilde{V} \ . \tag{11}$$

Now, suppose that $\{Z_2, Z_m\}$ and $\{Z_2', Z_m'\}$ are two sets of renormalization constants. They may correspond to two different renormalization scales μ and μ', respectively. Since there is a definite relationship between the bare and renormalized quantities, we know the relationship between the quantities renormalized by primed and unprimed Z's. Concretely, using (7) and (11), we transform the SDE, Eq. (8), and find that it does not change its form, so that the MR holds. Before further discussing the MR, we rewrite the SDE (8) using the conventional nonrelativistic *Ansatz* for the quark propagator S:

$$S^{-1}(k) = \not{k} - \Sigma(\vec{k}), \quad \Sigma(\vec{k}) = A(\vec{k})|\vec{k}| + B(\vec{k})\vec{\gamma} \cdot \vec{k}. \tag{12}$$

It is technically convenient for our purpose to express it through the functions $\varphi(\vec{k})$ and $\omega(\vec{k})$ in terms of which our SDE will later be written:

$$A(\vec{k}) = \omega(\vec{k}) \sin \varphi(\vec{k}), \quad B(\vec{k}) = \frac{1}{|\vec{k}|}\omega(\vec{k}) \cos \varphi(\vec{k}) - 1 \ . \tag{13}$$

It is also convenient to define the matrix ζ as

$$\zeta(\vec{k}) = \sin \frac{1}{2}\varphi(\vec{k}) - \hat{k} \cdot \vec{\gamma} \cos \frac{1}{2}\varphi(\vec{k}) \tag{14}$$

in order to write the inverse of the quark propagator as

$$S^{-1}(k) = k^0 \gamma^0 - \omega(\vec{k})\zeta^{-2}(\vec{k}) \ , \tag{15}$$

and the propagator itself as

$$S(k) = -\zeta(\vec{k})[\frac{\frac{1}{2}(1+\gamma^0)}{\omega(\vec{k}) - k^0 - \mathrm{i}\varepsilon} + \frac{\frac{1}{2}(1-\gamma^0)}{\omega(\vec{k}) + k^0 - \mathrm{i}\varepsilon}]\zeta(\vec{k}) \ . \tag{16}$$

Inserting (15) into the renormalized SDE (8) yields the following integral equations for ω and φ:

$$\omega(\vec{p})\sin\varphi(\vec{p}) - Z_m m + \frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\sin\varphi(\vec{k})\widetilde{V}(\vec{p}-\vec{k}) = 0 \; , \qquad (17a)$$

$$\omega(\vec{p})\cos\varphi(\vec{p}) - Z_2|\vec{p}| + \frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\cos\varphi(\vec{k})(\hat{k}\cdot\hat{p})\widetilde{V}(\vec{p}-\vec{k}) = 0. \qquad (17b)$$

Additionally, the equation $(Z_2 - 1)p^0 = 0$ arises, *i.e.*, $Z_2 = 1$. However, Eq. (17b) demands that Z_2 should be an infinite constant. In an influential paper, Adler and Davis [2] have resolved this contradiction by splitting Z_2 into two parts,

$$Z_2 \not{p} \to Z_0 p^0 \gamma^0 - Z\vec{p}\cdot\vec{\gamma} \; , \qquad (18)$$

and by setting $Z_0 = 1$. However, we have to remind ourselves that Z_2 is the wave–function renormalization constant, which defines the renormalization of the quark propagator, $S_0 = Z_2 S$. We are faced with a dilemma, namely, whether S should be renormalized with Z_0 or with Z. As one can expect, both possibilities change the form of the renormalized SDE Eq. (8). Obviously, the consistency of MR is violated because one has to split Z_2 into Z_0 and Z. If unprimed and primed renormalization constants correspond to two different renormalization scales μ and μ', respectively, this inconsistency shows that the invariance with respect to the changes of the renormalization scale μ is lost, and we cannot use the renormalization–group (RG) equations to relate results for one arbitrary scale μ to those for some other scale μ'. Nevertheless, as noted by Brown and Dorey [10], who explored the consistency of the MR of the SDE when various approximations are made, this does not mean that treatments that do fail such a consistency test are not useful; it may merely mean that it is more difficult to relate their solutions to real physics.

3. Standard renormalization prescription

In Ref. 2 quoted in the preceding section, Adler and Davis treated only the linear part of the potential numerically; *i.e.*, they did not perform concrete numerical calculations in the presence of the Coulombic part, which is the part causing UV divergences. However, they did discuss the renormalization problem theoretically and proposed two prescriptions for Z_2 subsequently used by authors who tried to solve the numerics with the Coulomb part present (*e.g.*, Refs. 4-7). Working in the quarkonium rest frame, Adler and Davis first noted that Z_2 was forced to split into two parts, Z_0 and Z, as given in Eq. (18) above. The first one, Z_0, had to obey the constraint $Z_0 = 1$, whereas Z had to be infinite. Their first prescription in our notation reads

$$Z - 1 = \frac{1}{2|\vec{p}|} \int \frac{d^3\vec{k}}{(2\pi)^3} \, (\hat{p} \cdot \hat{k}) \, \widetilde{V}_C(\vec{p} - \vec{k}) \; . \tag{19}$$

The crucial point emphasized by Adler and Davis [2] was that this Z corresponded to a momentum–dependent subtraction of UV divergences when the coupling constant α_s was running, *i.e.*, when α_s was momentum dependent. For that reason, they rejected the SDE with the Coulomb–like interaction of Ref. 1. For $\alpha_s =$ const, however, they noted that the infinite part of (19) defined a momentum–independent UV–subtraction, so it was judged acceptable. Indeed, this prescription has often been used thereafter (*e.g.*, Refs. 4-7). Also, in calculations with $m \neq 0$, the corresponding Z_m must be constructed [3,7]. Nevertheless, it is easy to see that such Z (and also Z_m of Ref. 7) still has a momentum–dependent finite part. We find it conceptually objectionable even if the UV infinities are successfully subtracted. The renormalization constants, the Z's, are of course not unique, as their finite parts differ from one renormalization prescription to another (and within the same renormalization scheme defined at some renormalization scale μ, Z's are different for different scales μ). However, these finite parts should, of course, also be constants as far as the dependence on the momentum variable is concerned. Otherwise, Z's do not just subtract constants, but can arbitrarily change the function dependence on momentum of the mathematical expression which is supposed to be renormalized by Z's!

The other possibility considered by Ref. 2 for the Z counterterm is given by their Eq. (2.17). It is important to note [3] that this expression is in fact given by our Eq. (19) in the limit $|\vec{p}| \to 0$. It is then evidently momentum–independent, since $|\vec{p}|$ is fixed to a specific value – zero. However, there is a subtlety which seems to have been unnoticed so far. After performing the angular integration on this second version of Z, one obtains an expression which is easily seen to diverge as the momentum–integration variable tends to zero. This means that the UV renormalization constant Z introduces a new IR divergence, which was not present in the original SDE! (Also, if $\alpha_s \neq$const but $\alpha_s = \alpha_s(k)$, this does not change the situation). However, the treatments using this Z [8] overlooked this divergence.

The aforementioned caveats motivate a reassessment of these normalization procedures. We start by commenting on the different vertex renormalizations of space and time components, as displayed in Eq. (18).

It is easy to see that this splitting of Z_2 is unavoidable if the usual nonrelativistic *Ansatz* (12) for the propagator S (or, equivalently, the self–energy Σ) is introduced, because then in addition to the renormalized SDE (17), the equation $(Z_2 - 1)p^0 = 0$ arises, *i.e.*, the constraint $Z_2 = 1$. On the other hand, (17b) requires infinite Z_2 because of $\widetilde{V}_C$ contained in $\widetilde{V}$. Thus, the noncovariant propagator *Ansatz* (12), a common simplification in the instantaneous–potential approach, unavoidably enforces the splitting of Z_2, and we do not have anything to add to the discussion of Adler and Davis [2] in this respect.

What we can do, however, is to mend the momentum dependence of Z's.

4. *Momentum–independent prescription*

A convenient renormalization prescription that determines Z and Z_m uniquely was given, *e.g.*, in Refs. 11, 12. The authors specified that the quark propagator $S(p)$, for a given spacelike $p^2 = -\mu^2$, agreed with free–field theory. We adopt this choice, adjusted for the instantaneous interaction and the noncovariant propagator *Ansatz* (12):

$$S^{-1}(p)|_{|\vec{p}|=\mu} = \not{p} - m \ . \tag{20}$$

Imposing (20) on the SDE (17) leads uniquely to the momentum–independent Z:

$$Z - 1 = \frac{1}{2\mu} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \cos\varphi(\vec{k})(\hat{\mu}\cdot\hat{k})\widetilde{V}_C(\vec{\mu}-\vec{k}) \ , \tag{21}$$

where $\vec{\mu}$ is a constant vector whose length μ defines the renormalization scale. This Z turns out to be quite adequate in a concrete bound–state calculation [3]. However, a similar expression for Z_m which follows (also uniquely) from Eq. (20) turns out to be unacceptable. If it is used in solving bound–state equations for, say, the pion [3], it leads to an incorrect chiral limit, where the pion mass $M_\pi \to 0$, not as $m \to 0$ but as $m \to m' = \omega_D(\mu)\sin\varphi_D(\mu) > 0$. Here the subscript D denotes the solutions φ and ω of the SDE (17) for $m = 0$.

This is because $\lim_{m\to 0} mZ_m \neq 0$ for Z_m stemming from (20), and this prevents the renormalized SDE (17) from reducing (for $m \to 0$) to what must be its massless version. Interestingly, all this can be traced back to the fact shown by Pagels [13] that the renormalization condition (20) precludes the presence of the dynamically generated mass term in the quark propagator $S(p)$.

However, it is possible to recover the proper chiral–limit behaviour by redefining the UV–finite part of Z_m:

$$Z_m - 1 = \frac{1}{m}\ \omega_D(\mu)\sin\varphi_D(\mu) + \frac{1}{2m}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\ \sin\varphi(\vec{k})\ \widetilde{V}_C(\vec{\mu}-\vec{k}) \ . \tag{22}$$

The second term is the old $Z_m - 1$ that would stem from (20), whereas the first term is the new finite term which is equivalent to the redefinition of the renormalization condition

$$S^{-1}(p)|_{|\vec{p}|\,=\,\mu} = [\not{p} - m - \omega_D(\vec{p})\sin\varphi_D(\vec{p})]_{|\vec{p}|\,=\,\mu} \ . \tag{23}$$

5. *Conclusion*

We have obtained a renormalization prescription which is momentum independent, in counterdistinction to the most standard one in this line of research. However, we point out the following: In a more detailed work [3] we simultaneously solved the SDE and the Bethe–Salpeter equation for the pion as $q\bar{q}$ bound by the funnel potential, where the UV–divergence was removed by our momentum–independent renormalization prescription. The pion mass, the constituent (dressed) quark mass, the pion decay constant f_π, and the decay width for $\pi^0 \to \gamma\gamma$ obtained in Ref. 3 are similar to those obtained by authors using the momentum–dependent renormalization constants [4-7]. We may conclude that, although the pairing model of the pion is not quantitatively successful regardless of which of the discussed renormalization procedures is used, they yield probably meaningful (model) results, as the latter are similar in different schemes. Of course, we prefer our momentum–independent scheme on conceptual grounds.

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IMPULSNO NEOVISNA RENORMALIZACIJA SCHWINGER–DYSONOVE JEDNADŽBE S TRENUTNOM INTERAKCIJOM

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Razmatramo problem koji se prilikom renormalizacije Schwinger–Dysonove jednadžbe susreće kada je jezgra međukvarkovske interakcije modelirana trenutnim potencijalom. Preciznije, Coulombov potencijal, potreban za opis gluonske izmjene na malim udaljenostima između kvarkova, dovodi do ultraljubičastih divergencija u Schwinger–Dysonovoj jednadžbi. Standardna postavka koja se dosad upotrebljavala za uklanjanje tih divergencija ima ozbiljan konceptualan problem: ona nije impulsno neovisna. U ovom radu predlažemo jednu novu impulsno neovisnu postavku, pri čemu posebno pazimo da bude očuvan ispravan kiralni limes za lagane pseudoskalarne mezone.